

CONSTRUCTION OF POINTS REALIZING THE REGULAR SYSTEMS OF WOLFGANG SCHMIDT AND LEONARD SUMMERER

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*On the occasion of Axel Thue's 150th birthday,
with special homage to Professor Wolfgang Schmidt on his 80th birthday.*

ABSTRACT. In a series of recent papers, W. M. Schmidt and L. Summerer developed a new theory by which they recover all major generic inequalities relating exponents of Diophantine approximation to a point in \mathbb{R}^n , and find new ones. Given a point in \mathbb{R}^n , they first show how most of its exponents of Diophantine approximation can be computed in terms of the successive minima of a parametric family of convex bodies attached to that point. Then they prove that these successive minima can in turn be approximated by a certain class of functions which they call (n, γ) -systems. In this way, they bring the whole problem to the study of these functions. To complete the theory, one would like to know if, conversely, given an (n, γ) -system, there exists a point in \mathbb{R}^n whose associated family of convex bodies has successive minima which approximate that function. In the present paper, we show that this is true for a class of functions which they call regular systems.

1. INTRODUCTION

Let $\xi_1, \dots, \xi_{n-1} \in \mathbb{R}$ for some integer $n \geq 2$. A basic problem in Diophantine approximation is to measure how well the point $(\xi_1, \dots, \xi_{n-1})$ can be approximated by rational points with common denominators below a given bound, and how small can integer linear combinations of $1, \xi_1, \dots, \xi_{n-1}$ be, given an upper bound on the absolute values of their coefficients. This gives rise to four classical exponents of approximation which are linked by the dualities of A. Y. Khintchine [5, 6] and V. Jarník [4]. In the case $n = 3$, M. Laurent achieved recently a complete description of the joint spectrum of these four exponents [7]. Such a description is still lacking in higher dimensions. However, N. Moshchevitin [8] recently found a new relation between these exponents in the case $n = 4$. Then, a second proof of it together with a proof of a “dual” relation was given by W. M. Schmidt and L. Summerer in [13] using their theory of parametric geometry of numbers. To show that both relations are best possible these authors ask for the existence of points in \mathbb{R}^4 satisfying certain conditions that we will recall below. The purpose of this note is to construct such points. For the interested reader, it can serve as an introduction to [9] where we construct points satisfying the fully general conditions provided by the theory of Schmidt and Summerer.

This wonderful theory, called parametric geometry of numbers by their authors, was developed first in dimension $n = 3$ in [11] and then for general dimension $n \geq 2$ in [12]. It

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provides a very precise description of the behavior of the successive minima of certain parametric families of convex bodies of \mathbb{R}^n . Here, the term *convex body* of \mathbb{R}^n refers to a compact 0-symmetric neighborhood \mathcal{C} of 0 in \mathbb{R}^n . We recall that, for $j = 1, \dots, n$, the j -th minimum $\lambda_j(\mathcal{C})$ of such a set is the smallest real number λ such that $\lambda\mathcal{C}$ contains at least j linearly independent elements of \mathbb{Z}^n . Clearly these minima form a monotone increasing sequence $\lambda_1(\mathcal{C}) \leq \dots \leq \lambda_n(\mathcal{C})$. Throughout this paper, we assume that the integer n is at least 2.

Let $\mathbf{x} \cdot \mathbf{y}$ denote the usual scalar product of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and let $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ denote the corresponding norm of a vector \mathbf{x} . For our purpose, we work with the families of convex bodies

$$\mathcal{C}_{\mathbf{u}}(Q) = \{ \mathbf{x} \in \mathbb{R}^n ; \|\mathbf{x}\| \leq Q, |\mathbf{x} \cdot \mathbf{u}| \leq Q^{-(n-1)} \} \quad (Q \geq 1).$$

where \mathbf{u} is a fixed unit vector of \mathbb{R}^n . These are essentially the polar reciprocal bodies to those considered in [12] but in view of the close relations linking the successive minima of a convex body to those of its polar reciprocal body, this makes very little difference. Besides its own fundamental intrinsic interest, a strong motivation for studying the successive minima of $\mathcal{C}_{\mathbf{u}}(Q)$ as functions of Q comes from the fact that, if we choose \mathbf{u} to be a multiple of $(1, \xi_1, \dots, \xi_{n-1})$, then the four exponents to which we alluded above can be computed directly from these functions (see [12, §1]), and the same holds for the intermediate exponents studied by Y. Bugeaud and M. Laurent in [1] (see also [2] and [10]). In fact, let

$$\Delta_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n ; x_1 \leq \dots \leq x_n\},$$

and consider the continuous map $\mathbf{L}_{\mathbf{u}}: [0, \infty) \rightarrow \Delta_n$ given by

$$L_{\mathbf{u}}(q) = (\log \lambda_1(\mathcal{C}_{\mathbf{u}}(e^q)), \dots, \log \lambda_n(\mathcal{C}_{\mathbf{u}}(e^q))) \quad (q \geq 0).$$

An approximation of $\mathbf{L}_{\mathbf{u}}$ with bounded difference suffices by far to compute these exponents.

In [12, §2], Schmidt and Summerer define, for each $\gamma \geq 0$ and each $a \geq 0$, the notion of an (n, γ) -system on the interval $[a, \infty)$. This is a continuous map $\mathbf{P}: [a, \infty) \rightarrow \mathbb{R}^n$ which satisfies a certain number of conditions which, although relatively easy to state, are somewhat difficult to analyze. The largest part of their paper deals with this issue. Here, since we essentially use the polar reciprocal bodies, the relevant notion for us is a dual one as in [13, §7]. However, for simplicity, we keep the same terminology. Then, modulo slight modifications, the authors establish in [12, §2] the existence of a constant $\gamma > 0$ and of an (n, γ) -system $\mathbf{P}: [0, \infty) \rightarrow \mathbb{R}^n$ such that $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded on $[0, \infty)$.

As shown in [12, §3], the behavior of an $(n, 0)$ -system is much easier to understand. In particular, such a map takes values in Δ_n . In [9], we show that, for each (n, γ) -system $\mathbf{P}: [0, \infty) \rightarrow \mathbb{R}^n$, there exist a real number $a \geq 0$ and an $(n, 0)$ -system $\mathbf{X}: [a, \infty) \rightarrow \Delta_n$ for which the difference $\mathbf{P} - \mathbf{X}$ is bounded on $[a, \infty)$. In view of the result of Schmidt and Summerer mentioned above, this means that, for any unit vector \mathbf{u} in \mathbb{R}^n , there exists an $(n, 0)$ -system $\mathbf{X}: [a, \infty) \rightarrow \Delta_n$ for which $\mathbf{L}_{\mathbf{u}} - \mathbf{X}$ is bounded on $[a, \infty)$. In [9], we also show that the converse is true namely that, for each $(n, 0)$ -system $\mathbf{X}: [a, \infty) \rightarrow \Delta_n$, there exists a unit vector \mathbf{u} of \mathbb{R}^n such that $\mathbf{L}_{\mathbf{u}} - \mathbf{X}$ is bounded on $[a, \infty)$. In particular, this

proves a conjecture of [12, §4] to the effect that all generic relations between exponents of approximation can be derived from the study of $(n, 0)$ -systems.

Our goal here is to construct unit vectors associated to a class of $(n, 0)$ -systems which is slightly more general than the *regular systems* of [13, §3]. To present this class of *quasi-regular $(n, 0)$ -systems*, we follow Schmidt and Summerer in [12, §3] and define the *combined graph* of a set of real valued functions defined on an interval I to be the union of their graphs in $I \times \mathbb{R}$. For a function $\mathbf{P} = (P_1, \dots, P_n): [a, \infty) \rightarrow \Delta_n$, and a sub-interval I of $[a, \infty)$, we define the *combined graph of \mathbf{P} above I* to be the combined graph of its components P_1, \dots, P_n restricted to I . If P is continuous and if the real numbers $q \geq a$ at which $P_1(q), \dots, P_n(q)$ are not all distinct form a discrete subset of $[a, \infty)$, then the map \mathbf{P} is uniquely determined by its combined graph over the full interval $[a, \infty)$. We also denote by $\Phi_n: \mathbb{R}^n \rightarrow \Delta_n$ the continuous map which lists the coordinates of a point in monotone increasing order.

Definition 1.1. A *quasi-regular $(n, 0)$ -system* is a continuous function $\mathbf{P}: [a, \infty) \rightarrow \Delta_n$ for which there exists an unbounded strictly increasing sequence of positive real numbers $(X_i)_{i \geq 1}$ such that, upon defining

$$q_i = (X_i + \dots + X_{i+n-1})/n \quad (i \geq 1),$$

we have $a = q_1$ and, for each $i \geq 1$,

$$(1.1) \quad \mathbf{P}(q) = \Phi_n(X_i + n(q - q_i) - q, X_{i+1} - q, \dots, X_{i+n-1} - q) \quad (q_i \leq q \leq q_{i+1}).$$

If, for some $\delta > 0$, we also have $X_{i+1} \geq X_i + \delta$ for each $i \geq 1$, then we say that \mathbf{P} *has mesh at least δ* . If there exists $\rho > 1$ such that $X_{i+1} = \rho X_i$ for each $i \geq 1$, then we say that \mathbf{P} is *regular*.

Since $X_i + n(q_{i+1} - q_i) - q_{i+1} = X_{i+n} - q_{i+1}$, the condition (1.1) implies that

$$\mathbf{P}(q_i) = (X_i - q_i, \dots, X_{i+n-1} - q_i) \quad \text{and} \quad \mathbf{P}(q_{i+1}) = (X_{i+1} - q_{i+1}, \dots, X_{i+n} - q_{i+1}).$$

Therefore, upon writing $\mathbf{P} = (P_1, \dots, P_n)$, it is equivalent to asking that the combined graph of \mathbf{P} above $[q_i, q_{i+1}]$ consists of one line segment of slope $n - 1$ joining $(q_i, P_1(q_i))$ to $(q_{i+1}, P_n(q_{i+1}))$, together with $n - 1$ distinct line segments of slope -1 joining $(q_i, P_{j+1}(q_i))$ to $(q_{i+1}, P_j(q_{i+1}))$ for $j = 1, \dots, n - 1$.

The above remark shows in particular that any choice of $0 < X_1 < X_2 < \dots$ with $\lim_{i \rightarrow \infty} X_i = \infty$ gives rise to a continuous map $\mathbf{P}: [q_1, \infty) \rightarrow \Delta_n$ satisfying (1.1) for each $i \geq 1$. It also implies that, in turn, such a map \mathbf{P} uniquely determines the sequence $(X_i)_{i \geq 1}$ because the local minima of its first component P_1 are the points $(q_i, P_1(q_i)) = (q_i, X_i - q_i)$ ($i \geq 1$). This is illustrated on Figure 1 below which shows in solid lines the combined graph of a quasi-regular $(4, 0)$ -system over an interval $[q_1, q_5]$.

A general $(n, 0)$ -system also comes with a partition of its domain into subintervals above which its combined graph consists of a line segment of slope $n - 1$ and $n - 1$ line segments of slope -1 , but there is more flexibility in the way in which these line segments connect

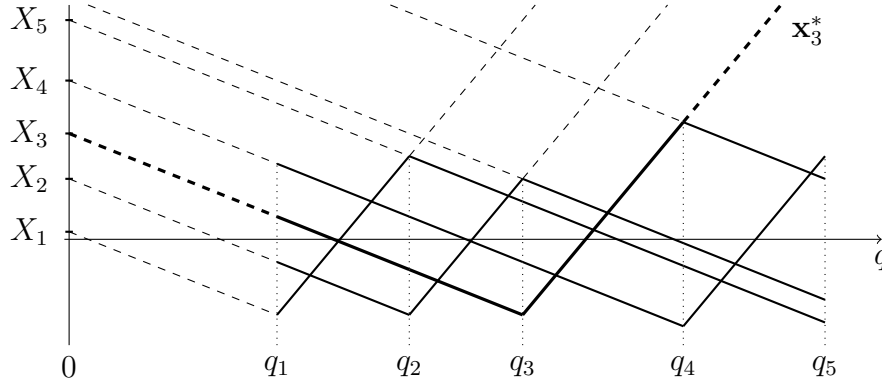


FIGURE 1. Example of combined graph of a quasi-regular $(4, 0)$ -system over an interval $[q_1, q_5]$, with the trajectory of an ideal point \mathbf{x}_3^* enlightened.

the points above the left and the right end-points of the subintervals. In the case of a quasi-regular $(n, 0)$ -system, the line segments of slope $n - 1$ always connect the lowest point on the left to the highest point on the right.

The main result of this paper is the following statement where $\|\cdot\|_\infty$ stands for the maximum norm.

Theorem 1.2. *Let $\mathbf{P}: [q_1, \infty) \rightarrow \Delta_n$ be a quasi-regular $(n, 0)$ -system with mesh at least $\log 4$. Then there exists a unit vector \mathbf{u} of \mathbb{R}^n such that*

$$\|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_\infty \leq 2n^2 \quad (q \geq q_1).$$

To say a word about the proof, recall that each convex body \mathcal{C} of \mathbb{R}^n induces a *distance function* on \mathbb{R}^n . It is the map from \mathbb{R}^n to $[0, \infty)$ which assigns to each point \mathbf{x} of \mathbb{R}^n the smallest real number $\lambda \geq 0$, denoted $\lambda(\mathbf{x}, \mathcal{C})$, such that $\mathbf{x} \in \lambda\mathcal{C}$ (see [3, §1.3]). Usually, \mathcal{C} is fixed and \mathbf{x} varies. Here, the situation is reversed. The point $\mathbf{x} \in \mathbb{R}^n$ is fixed and we let the convex body \mathcal{C} vary within the family $\mathcal{C}_{\mathbf{u}}(Q)$ with $Q \geq 1$, for some unit vector \mathbf{u} of \mathbb{R}^n . In view of the definition of $\mathcal{C}_{\mathbf{u}}(Q)$, we have

$$(1.2) \quad \lambda(\mathbf{x}, \mathcal{C}_{\mathbf{u}}(Q)) = \max \{ \|\mathbf{x}\|Q^{-1}, |\mathbf{x} \cdot \mathbf{u}|Q^{n-1} \} \quad (Q \geq 1).$$

Suppose that the coordinates of \mathbf{u} are linearly independent over \mathbb{Q} and that $\mathbf{x} \in \mathbb{Z}^n \setminus \{0\}$. Then, we have $0 < |\mathbf{x} \cdot \mathbf{u}| < \|\mathbf{x}\|$ and we define a map $L_{\mathbf{x}}: [0, \infty) \rightarrow \mathbb{R}$ by

$$\begin{aligned} L_{\mathbf{x}}(q) &:= L(\mathbf{x}, q) := \log \lambda(\mathbf{x}, \mathcal{C}_{\mathbf{u}}(e^q)) \\ &= \max \{ \log \|\mathbf{x}\| - q, \log |\mathbf{x} \cdot \mathbf{u}| + (n-1)q \} \quad (q \geq 0). \end{aligned}$$

Its graph is a polygon with two sides: a line segment of slope -1 followed by an half-line with slope $n - 1$. The function $L_{\mathbf{x}}$ is continuous and has a local minimum at the point where its graph changes slope from -1 to $n - 1$. Although \mathbf{x} is fixed, we say that $L_{\mathbf{x}}$, or its graph, represents the *trajectory* of the point \mathbf{x} with respect to the varying family of convex bodies $\mathcal{C}_{\mathbf{u}}(Q)$. Clearly, this trajectory is uniquely determined by its local minimum. It is

not difficult to show that the combined graph of $\mathbf{L}_{\mathbf{u}}$ above any compact interval is covered by the trajectories of finitely many non-zero integer points (see [11, §4]).

Now, let $\mathbf{P}: [q_1, \infty) \rightarrow \Delta_n$ be a quasi-regular $(n, 0)$ -system. In the notation of Definition 1.1, we can imagine its combined graph covered by the trajectories of a sequence of “ideal points” \mathbf{x}_i^* having local minima at $(q_i, P_1(q_i))$. Figure 1 shows the trajectory of such an ideal point \mathbf{x}_3^* . In general, we cannot hope for such points to exist. Instead, we construct a sequence $(\mathbf{x}_i)_{i \geq 1}$ of integer points and a unit vector \mathbf{u} such that, for each $i \geq 1$, the trajectory of \mathbf{x}_i is close to ideal and moreover the n -tuple $(\mathbf{x}_i, \dots, \mathbf{x}_{i+n-1})$ is a basis of \mathbb{Z}^n . In practice, the vector \mathbf{u} is also constructed as a limit of unit vectors \mathbf{u}_i where \mathbf{u}_i is perpendicular to $\mathbf{x}_i, \dots, \mathbf{x}_{i+n-2}$ for each $i \geq 1$. Then, it suffices to choose the sequence $(\mathbf{x}_i)_{i \geq 1}$ so that the trajectory of \mathbf{x}_i with respect to the family $\mathcal{C}_{\mathbf{u}_{i+1}}(Q)$ is close to ideal. To this end, we require \mathbf{P} to have mesh at least $\log 4$. This allows us to control appropriately the norms of the points \mathbf{x}_i as well as the angles that they make with respect to certain subspaces.

2. ALMOST ORTHOGONAL SEQUENCES

For each $k = 1, \dots, n$, we endow $\bigwedge^k \mathbb{R}^n$ with the Euclidean space structure characterized by the property that, for any orthonormal basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ of \mathbb{R}^n , the products $\mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_k}$ with $1 \leq j_1 < \dots < j_k \leq n$ form an orthonormal basis of $\bigwedge^k \mathbb{R}^n$. We denote by $\|\mathbf{p}\|$ the associated norm of an element \mathbf{p} of $\bigwedge^k \mathbb{R}^n$. We also denote by $\bigwedge^k \mathbb{Z}^n$ the lattice of $\bigwedge^k \mathbb{R}^n$ of co-volume 1 spanned by the products $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k$ with $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{Z}^n$.

The projective distance between two non-zero points \mathbf{x}, \mathbf{y} of \mathbb{R}^n is

$$\text{dist}(\mathbf{x}, \mathbf{y}) := \frac{\|\mathbf{x} \wedge \mathbf{y}\|}{\|\mathbf{x}\| \|\mathbf{y}\|}.$$

It depends only on the classes of \mathbf{x} and \mathbf{y} in $\mathbb{P}^{n-1}(\mathbb{R})$ and represents the sine of the angle between the one-dimensional subspaces of \mathbb{R}^n spanned by \mathbf{x} and \mathbf{y} . This function induces a metric on $\mathbb{P}^{n-1}(\mathbb{R})$ (satisfying the triangle inequality) and $\mathbb{P}^{n-1}(\mathbb{R})$ is complete with respect to that metric.

Given a point \mathbf{x} of \mathbb{R}^n and a subspace U of \mathbb{R}^n , we denote by U^\perp the orthogonal complement of U and by $\text{proj}_U(\mathbf{x})$ the orthogonal projection of \mathbf{x} on U . If \mathbf{x} is non-zero, we also define

$$\text{dist}(\mathbf{x}, U) := \frac{\|\text{proj}_{U^\perp}(\mathbf{x})\|}{\|\mathbf{x}\|}.$$

The next lemma connects the two notions of distance.

Lemma 2.1. *If \mathbf{x} is a non-zero point of \mathbb{R}^n , and if U is a non-zero proper subspace of \mathbb{R}^n with basis $(\mathbf{y}_1, \dots, \mathbf{y}_k)$, then*

$$\text{dist}(\mathbf{x}, U) = \frac{\|\mathbf{x} \wedge \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k\|}{\|\mathbf{x}\| \|\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k\|} = \min\{\text{dist}(\mathbf{x}, \mathbf{y}) ; \mathbf{y} \in U \setminus \{0\}\}.$$

Proof. The first formula follows from the definition using

$$\|\mathbf{x} \wedge \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k\| = \|\text{proj}_{U^\perp}(\mathbf{x}) \wedge \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k\| = \|\text{proj}_{U^\perp}(\mathbf{x})\| \|\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_k\|.$$

It implies in particular that $\text{dist}(\mathbf{x}, \mathbf{y}) = \text{dist}(\mathbf{x}, \langle \mathbf{y} \rangle_{\mathbb{R}})$ for any $\mathbf{y} \in \mathbb{R}^n \setminus \{0\}$. To prove the second equality of the lemma, we first note that, for any subspace V of U , we have $\text{proj}_{U^\perp}(\mathbf{x}) = \text{proj}_{U^\perp}(\text{proj}_{V^\perp}(\mathbf{x}))$ and so $\text{dist}(\mathbf{x}, U) \leq \text{dist}(\mathbf{x}, V)$. In particular, this implies that $\text{dist}(\mathbf{x}, U) \leq \text{dist}(\mathbf{x}, \mathbf{y})$ for any $\mathbf{y} \in U \setminus \{0\}$. If $\mathbf{x} \notin U^\perp$, then $\mathbf{y} := \text{proj}_U(\mathbf{x})$ is a non-zero element of U with $\text{dist}(\mathbf{x}, U) = \text{dist}(\mathbf{x}, \mathbf{y})$ because \mathbf{x} has the same orthogonal projection on U^\perp as on $\langle \mathbf{y} \rangle_{\mathbb{R}}^\perp$. Thus the second equality holds in that case. If $\mathbf{x} \in U^\perp$, then it still holds because $\text{dist}(\mathbf{x}, U) = 1 = \text{dist}(\mathbf{x}, \mathbf{y})$ for any $\mathbf{y} \in U \setminus \{0\}$. \square

Definition 2.2. We say that a sequence $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ of vectors of \mathbb{R}^n is *almost orthogonal* if it is linearly independent and if

$$\text{dist}(\mathbf{x}_j, \langle \mathbf{x}_1, \dots, \mathbf{x}_{j-1} \rangle_{\mathbb{R}}) \geq 1/2 \quad (2 \leq j \leq k).$$

By Lemma 2.1, it follows that any subsequence of an almost orthogonal sequence is almost orthogonal. Moreover, if $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ is almost orthogonal, then

$$\|\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k\| = \|\mathbf{x}_1\| \prod_{j=2}^k (\|\mathbf{x}_j\| \text{dist}(\mathbf{x}_j, \langle \mathbf{x}_1, \dots, \mathbf{x}_{j-1} \rangle_{\mathbb{R}})) \geq 2^{-(k-1)} \|\mathbf{x}_1\| \dots \|\mathbf{x}_k\|.$$

Note that in [9], we use a stronger notion of almost orthogonality.

We say that an element \mathbf{x} of \mathbb{Z}^n is *primitive* if it is non-zero and if its coordinates are relatively prime as a set. More generally, we say that a k -tuple $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ of elements of \mathbb{Z}^n is *primitive* if $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k$ is non-zero and if its coordinates with respect to a basis of $\bigwedge^k \mathbb{Z}^n$ are relatively prime. This condition is equivalent to asking that $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ can be extended to a basis $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of \mathbb{Z}^n . In particular, it requires that $1 \leq k \leq n$.

Finally, we say that a non-zero subspace U of \mathbb{R}^n is *defined over* \mathbb{Q} if it is spanned by elements of \mathbb{Q}^n . Following Schmidt in [10], we then define the *height* of U by

$$H(U) = \|\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k\|$$

where $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ is any basis of $U \cap \mathbb{Z}^n$. This is independent of the choice of the basis. The next result summarizes some of the above considerations.

Lemma 2.3. *Let $(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})$ be an almost orthogonal primitive $(n-1)$ -tuple of points of \mathbb{Z}^n and let $U := \langle \mathbf{x}_1, \dots, \mathbf{x}_{n-1} \rangle_{\mathbb{R}}$. Then, we have*

$$2^{-(n-2)} \|\mathbf{x}_1\| \dots \|\mathbf{x}_{n-1}\| \leq H(U) \leq \|\mathbf{x}_1\| \dots \|\mathbf{x}_{n-1}\|.$$

We conclude this section with a particular construction of almost orthogonal sequences. It will serve as the initial step for a recursive construction of integer points in the next section.

Lemma 2.4. *Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ denote the canonical basis of \mathbb{Z}^n and let $B_1, \dots, B_{n-1} \in \mathbb{Z}$ with $B_i \geq 2^{i-1}$ for $i = 1, \dots, n-1$. Set*

$$\mathbf{x}_i = B_i \mathbf{e}_i + \mathbf{e}_{i+1} \quad (i = 1, \dots, n-1).$$

Then $(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})$ is an almost orthogonal primitive $(n-1)$ -tuple of integer points.

Proof. We first note that $(\mathbf{e}_1, \mathbf{x}_1, \dots, \mathbf{x}_{n-1})$ is a basis of \mathbb{Z}^n and therefore $(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})$ is primitive. Let k be an integer with $2 \leq k \leq n-1$. Since

$$\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k \wedge \mathbf{e}_{k+1} = B_1 \dots B_k \mathbf{e}_1 \wedge \dots \wedge \mathbf{e}_{k+1},$$

we must have $\|\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_k\| \geq B_1 \dots B_k$. As we also have

$$\|\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_{k-1}\| \|\mathbf{x}_k\| \leq \prod_{i=1}^k \|\mathbf{x}_i\| = \prod_{i=1}^k \sqrt{1 + B_i^2} \leq \prod_{i=1}^k \left(B_i \exp\left(\frac{1}{2B_i^2}\right) \right) \leq 2 \prod_{i=1}^k B_i,$$

we conclude from Lemma 2.1 that $\text{dist}(\mathbf{x}_k, \langle \mathbf{x}_1, \dots, \mathbf{x}_{k-1} \rangle_{\mathbb{R}}) \geq 1/2$. This shows that the sequence $(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})$ is almost orthogonal. \square

3. A RECURSIVE CONSTRUCTION OF POINTS

The next lemma is the key to a recursive construction of points in \mathbb{Z}^n which is at the heart of the proof of our main theorem.

Lemma 3.1. *Let $(\mathbf{y}_1, \dots, \mathbf{y}_{n-1})$ be an almost orthogonal primitive $(n-1)$ -tuple of points of \mathbb{Z}^n and let A be a real number with $A \geq 2 + \|\mathbf{y}_1\| + \dots + \|\mathbf{y}_{n-1}\|$. Then, there exists a point $\mathbf{y}_n \in \mathbb{Z}^n$ with the following properties*

- 1) $A \leq \|\mathbf{y}_n\| \leq 2A$,
- 2) $(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n)$ is a basis of \mathbb{Z}^n ,
- 3) $(\mathbf{y}_2, \dots, \mathbf{y}_n)$ is almost orthogonal,
- 4) if \mathbf{u} is a unit vector perpendicular to $U := \langle \mathbf{y}_1, \dots, \mathbf{y}_{n-1} \rangle_{\mathbb{R}}$, and if \mathbf{u}' is a unit vector perpendicular to $U' := \langle \mathbf{y}_2, \dots, \mathbf{y}_n \rangle_{\mathbb{R}}$, then

$$\text{dist}(\mathbf{u}, \mathbf{u}') \leq \frac{1}{A H(U)} \quad \text{and} \quad |\mathbf{y}_1 \cdot \mathbf{u}'| = \frac{1}{H(U')}.$$

Proof. Let U and \mathbf{u} be as in the condition 4). We define $V = \langle \mathbf{y}_2, \dots, \mathbf{y}_{n-1} \rangle_{\mathbb{R}}$, and choose a unit vector \mathbf{v} of U which is perpendicular to V . Then (\mathbf{u}, \mathbf{v}) is an orthonormal basis for V^\perp .

The hyperplane $H(U)^{-1}\mathbf{u} + U$ is a closest translate of U which contains a point of \mathbb{Z}^n not in U . For any point \mathbf{y} of this hyperplane, we have $|\det(\mathbf{y}_1, \dots, \mathbf{y}_{n-1}, \mathbf{y})| = 1$ and there exist $\epsilon_1, \dots, \epsilon_{n-1} \in [-1/2, 1/2]$ such that

$$\mathbf{y} + \epsilon_1 \mathbf{y}_1 + \dots + \epsilon_{n-1} \mathbf{y}_{n-1} \in \mathbb{Z}^n.$$

We apply this to the point $\mathbf{y} = H(U)^{-1}\mathbf{u} + (3/2)A\mathbf{v}$. This yields an integer point

$$\mathbf{y}_n := \frac{1}{H(U)}\mathbf{u} + \frac{3}{2}A\mathbf{v} + \epsilon_1 \mathbf{y}_1 + \dots + \epsilon_{n-1} \mathbf{y}_{n-1} \in \mathbb{Z}^n$$

for which $(\mathbf{y}_1, \dots, \mathbf{y}_n)$ is a basis of \mathbb{Z}^n because $|\det(\mathbf{y}_1, \dots, \mathbf{y}_n)| = 1$. Since $H(U) \geq 1$, we also find

$$\left\| \mathbf{y}_n - \frac{3}{2}A\mathbf{v} \right\| \leq 1 + \frac{1}{2}(\|\mathbf{y}_1\| + \dots + \|\mathbf{y}_{n-1}\|) \leq \frac{A}{2}$$

and thus $A \leq \|\mathbf{y}_n\| \leq 2A$. This shows that the conditions 1) and 2) hold.

Since the orthogonal projection of \mathbf{y}_n on V^\perp has norm at least

$$(3.1) \quad |\mathbf{y}_n \cdot \mathbf{v}| = \left| \frac{3}{2}A + \epsilon_1 \mathbf{y}_1 \cdot \mathbf{v} \right| \geq \frac{3}{2}A - \frac{1}{2}\|\mathbf{y}_1\| \geq A,$$

we find that

$$\text{dist}(\mathbf{y}_n, \langle \mathbf{y}_2, \dots, \mathbf{y}_{n-1} \rangle_{\mathbb{R}}) = \text{dist}(\mathbf{y}_n, V) = \frac{\|\text{proj}_{V^\perp}(\mathbf{y}_n)\|}{\|\mathbf{y}_n\|} \geq \frac{A}{\|\mathbf{y}_n\|} \geq \frac{1}{2}.$$

We also note that

$$\text{dist}(\mathbf{y}_i, \langle \mathbf{y}_2, \dots, \mathbf{y}_{i-1} \rangle_{\mathbb{R}}) \geq \text{dist}(\mathbf{y}_i, \langle \mathbf{y}_1, \dots, \mathbf{y}_{i-1} \rangle_{\mathbb{R}}) \geq \frac{1}{2} \quad (3 \leq i \leq n-1)$$

because $(\mathbf{y}_1, \dots, \mathbf{y}_{n-1})$ is almost orthogonal. Thus $(\mathbf{y}_2, \dots, \mathbf{y}_n)$ is almost orthogonal as well, and so the condition 3) holds.

Let $U' := \langle \mathbf{y}_2, \dots, \mathbf{y}_n \rangle_{\mathbb{R}}$ and let \mathbf{u}' be a unit vector perpendicular to U' . Since $V \subset U'$, we have $\mathbf{u}' \in V^\perp$ and so we can write

$$\mathbf{u}' = a\mathbf{u} + b\mathbf{v}$$

for some $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$. Since $\mathbf{y}_n \in U'$, we have $0 = \mathbf{y}_n \cdot \mathbf{u}'$ and so

$$|b| = |a| \frac{|\mathbf{y}_n \cdot \mathbf{u}|}{|\mathbf{y}_n \cdot \mathbf{v}|} \leq \frac{|\mathbf{y}_n \cdot \mathbf{u}|}{A} = \frac{1}{AH(U)}$$

where the middle inequality uses (3.1) and $|a| \leq 1$. We conclude that

$$\text{dist}(\mathbf{u}, \mathbf{u}') = \|\mathbf{u} \wedge \mathbf{u}'\| = \|b\mathbf{u} \wedge \mathbf{v}\| = |b| \leq \frac{1}{AH(U)}.$$

Finally, we find that

$$1 = |\det(\mathbf{y}_1, \dots, \mathbf{y}_n)| = |\mathbf{y}_1 \cdot \mathbf{u}'| \|\mathbf{y}_2 \wedge \dots \wedge \mathbf{y}_n\| = |\mathbf{y}_1 \cdot \mathbf{u}'| H(U')$$

and so $|\mathbf{y}_1 \cdot \mathbf{u}'| = H(U')^{-1}$. □

Proposition 3.2. *Let $(A_i)_{i \geq 1}$ be a sequence of real numbers with $A_1 \geq 1$ and $A_{i+1} \geq 4A_i$ for each $i \geq 1$. Then there exist a sequence of points $(\mathbf{x}_i)_{i \geq 1}$ in \mathbb{Z}^n and a unit vector \mathbf{u} of \mathbb{R}^n which, for each index $i \geq 1$, fulfil the following conditions:*

- 1) $(\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+n-1})$ is a basis of \mathbb{Z}^n ,
- 2) $A_i \leq \|\mathbf{x}_i\| \leq 2A_i$,
- 3) $2^{-n} \leq |\mathbf{x}_i \cdot \mathbf{u}| A_{i+1} \cdots A_{i+n-1} \leq 2^n$.

Proof. We first construct an almost orthogonal primitive $(n-1)$ -tuple $(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})$ as in Lemma 2.4 using $B_1 = \lceil A_1 \rceil, \dots, B_{n-1} = \lceil A_{n-1} \rceil$. Then these points satisfy $A_i \leq \|\mathbf{x}_i\| \leq 2A_i$ for $i = 1, \dots, n-1$. We set

$$U_1 = \langle \mathbf{x}_1, \dots, \mathbf{x}_{n-1} \rangle_{\mathbb{R}}$$

and denote by \mathbf{u}_1 a unit vector of \mathbb{R}^n orthogonal to U_1 . Then, using the fact that

$$2 + 2A_i + \dots + 2A_{i+n-2} \leq 2(1 + A_1 + \dots + A_{i+n-2}) \leq A_{i+n-1} \quad (i \geq 1),$$

Lemma 3.1 allows us to construct recursively, for each $i \geq 1$, an additional integer point \mathbf{x}_{i+n-1} , an additional $(n-1)$ -dimensional vector subspace U_{i+1} and an additional unit vector \mathbf{u}_{i+1} with the following properties

- 1) $A_{i+n-1} \leq \|\mathbf{x}_{i+n-1}\| \leq 2A_{i+n-1}$,
- 2) $(\mathbf{x}_i, \dots, \mathbf{x}_{i+n-1})$ is a basis of \mathbb{Z}^n ,
- 3) $(\mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+n-1})$ is almost orthogonal,
- 4) $U_{i+1} = \langle \mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+n-1} \rangle_{\mathbb{R}}$ and \mathbf{u}_{i+1} is perpendicular to U_{i+1} ,
- 5) $\text{dist}(\mathbf{u}_i, \mathbf{u}_{i+1}) \leq A_{i+n-1}^{-1} H(U_i)^{-1}$,
- 6) $|\mathbf{x}_i \cdot \mathbf{u}_{i+1}| = H(U_{i+1})^{-1}$.

Thanks to Lemma 2.3, we have

$$2^{-(n-2)} \|\mathbf{x}_i\| \cdots \|\mathbf{x}_{i+n-2}\| \leq H(U_i) \leq \|\mathbf{x}_i\| \cdots \|\mathbf{x}_{i+n-2}\| \quad (i \geq 1),$$

and therefore

$$(3.2) \quad 2^{-(n-2)} A_i \cdots A_{i+n-2} \leq H(U_i) \leq 2^{n-1} A_i \cdots A_{i+n-2} \quad (i \geq 1).$$

In view of the growth of the sequence $(A_i)_{i \geq 1}$, this implies that $H(U_{i+1}) \geq 2H(U_i)$ for each $i \geq 1$. Then, using 5), we deduce that the image of $(\mathbf{u}_i)_{i \geq 1}$ in $\mathbb{P}^{n-1}(\mathbb{R})$ converges to the class of a unit vector \mathbf{u} with

$$\text{dist}(\mathbf{u}_i, \mathbf{u}) \leq \sum_{j=i}^{\infty} \text{dist}(\mathbf{u}_j, \mathbf{u}_{j+1}) \leq \sum_{j=i}^{\infty} \frac{1}{A_{j+n-1} H(U_j)} \leq \frac{2}{A_{i+n-1} H(U_i)} \quad (i \geq 1).$$

Fix an index $i \geq 1$. Upon replacing \mathbf{u}_{i+1} by $-\mathbf{u}_{i+1}$ if necessary, we may assume that $\mathbf{u}_{i+1} \cdot \mathbf{u} \geq 0$. Then, the above estimate yields

$$\begin{aligned} |\mathbf{x}_i \cdot \mathbf{u} - \mathbf{x}_i \cdot \mathbf{u}_{i+1}| &\leq \|\mathbf{x}_i\| \|\mathbf{u} - \mathbf{u}_{i+1}\| \\ &\leq 2\|\mathbf{x}_i\| \text{dist}(\mathbf{u}, \mathbf{u}_{i+1}) \leq \frac{4\|\mathbf{x}_i\|}{A_{i+n} H(U_{i+1})} \leq \frac{1}{2H(U_{i+1})} \end{aligned}$$

since $A_{i+n} \geq 4^n A_i \geq 8\|\mathbf{x}_i\|$. In view of 6), this implies that

$$\frac{1}{2H(U_{i+1})} \leq |\mathbf{x}_i \cdot \mathbf{u}| \leq \frac{2}{H(U_{i+1})}.$$

Using the estimates for $H(U_{i+1})$ given by (3.2), this shows that the third condition of the proposition is satisfied. \square

In view of the formula (1.2) for $\lambda(\mathbf{x}, \mathcal{C}_{\mathbf{u}}(Q))$, the estimates of the proposition yield the following result.

Corollary 3.3. *Let the notation be as in the proposition. For each integer $i \geq 1$ and each real number $Q \geq 1$, we have*

$$2^{-n} \frac{A_i}{Q} \max \left\{ 1, \frac{Q}{Q_i} \right\}^n \leq \lambda(\mathbf{x}_i, \mathcal{C}_{\mathbf{u}}(Q)) \leq 2^n \frac{A_i}{Q} \max \left\{ 1, \frac{Q}{Q_i} \right\}^n.$$

where $Q_i = (A_i \cdots A_{i+n-1})^{1/n}$.

4. PROOF OF THE MAIN THEOREM

To deduce our main theorem from Proposition 3.2 and its corollary, we simply use the following well-known principle.

Lemma 4.1. *Let \mathcal{C} be a convex body of \mathbb{R}^n and let $\mathbf{y}_1, \dots, \mathbf{y}_n$ be linearly independent points of \mathbb{Z}^n . Suppose that*

$$(4.1) \quad \lambda(\mathbf{y}_1, \mathcal{C}) \cdots \lambda(\mathbf{y}_n, \mathcal{C}) \text{vol}(\mathcal{C}) \leq B$$

for some real number B . Then, we have

$$(\lambda_1(\mathcal{C}), \dots, \lambda_n(\mathcal{C})) \leq \Phi_n(\lambda(\mathbf{y}_1, \mathcal{C}), \dots, \lambda(\mathbf{y}_n, \mathcal{C})) \leq \frac{n!B}{2^n}(\lambda_1(\mathcal{C}), \dots, \lambda_n(\mathcal{C})),$$

where the inequality is meant component-wise.

Proof. Choose a permutation $\sigma \in S_n$ such that $\lambda(\mathbf{y}_{\sigma(1)}, \mathcal{C}) \leq \dots \leq \lambda(\mathbf{y}_{\sigma(n)}, \mathcal{C})$. By definition of the successive minima, we have $\lambda_j(\mathcal{C}) \leq \lambda(\mathbf{y}_{\sigma(j)}, \mathcal{C})$ for $j = 1, \dots, n$. As Minkowski's second convex body theorem gives

$$\frac{2^n}{n!} \leq \lambda_1(\mathcal{C}) \cdots \lambda_n(\mathcal{C}) \text{vol}(\mathcal{C}),$$

comparison with (4.1) yields

$$\lambda_j(\mathcal{C}) \leq \lambda(\mathbf{y}_{\sigma(j)}, \mathcal{C}) \leq \frac{n!B}{2^n} \lambda_j(\mathcal{C}) \quad (1 \leq j \leq n). \quad \square$$

Proof of Theorem 1.2. Let $(X_i)_{i \geq 1}$ and $(q_i)_{i \geq 1}$ be as in Definition 1.1, for the given quasi-regular $(n, 0)$ -system \mathbf{P} . We define

$$A_i := \exp(X_i) \quad (i \geq 1).$$

For this choice of parameters, we select a sequence of integer points $(\mathbf{x}_i)_{i \geq 1}$ and a unit vector \mathbf{u} which satisfy the conclusion of Proposition 3.2. We also define

$$L(\mathbf{x}_i, q) := \log \lambda(\mathbf{x}_i, \mathcal{C}_{\mathbf{u}}(e^q)) \quad (q \geq 0, i \geq 1).$$

Since $\exp(q_j) = (A_j \cdots A_{j+n-1})^{1/n}$ for each $j \geq 1$, Corollary 3.3 yields

$$(4.2) \quad |L(\mathbf{x}_j, q) - X_j - n \max\{0, q - q_j\} + q| \leq n \log(2) \quad (q \geq 0, j \geq 1).$$

To show that the vector \mathbf{u} has the required property, we fix an integer $i \geq 1$ and a real number $q \in [q_i, q_{i+1}]$. The points $\mathbf{x}_i, \dots, \mathbf{x}_{i+n-1}$ form a basis of \mathbb{Z}^n and, since $q_i \leq q \leq q_{i+1}$, the estimates (4.2) show that they satisfy

$$|L(\mathbf{x}_i, q) - X_i - n(q - q_i) + q| \leq n \log 2,$$

$$|L(\mathbf{x}_{i+1}, q) - X_{i+1} + q| \leq n \log 2,$$

...

$$|L(\mathbf{x}_{i+n-1}, q) - X_{i+n-1} + q| \leq n \log 2.$$

On one hand, these inequalities give

$$\|\mathbf{P}(q) - \Phi_n(L(\mathbf{x}_i, q), \dots, L(\mathbf{x}_{i+n-1}, q))\|_\infty \leq n \log 2.$$

On the other hand, since $\text{vol}(\mathcal{C}(e^q)) \leq 2^n$, they also lead to

$$L(\mathbf{x}_i, q) + \dots + L(\mathbf{x}_{i+n-1}, q) + \log \text{vol}(\mathcal{C}(e^q)) \leq (n^2 + n) \log 2$$

which, by Lemma 4.1, implies that

$$\|\mathbf{L}_u(q) - \Phi_n(L(\mathbf{x}_i, q), \dots, L(\mathbf{x}_{i+n-1}, q))\|_\infty \leq (n^2 + n) \log 2 + \log(n!/2^n).$$

This gives $\|\mathbf{P}(q) - \mathbf{L}_u(q)\|_\infty \leq (n^2 + n) \log(2) + \log(n!) \leq 2n^2$, as requested. \square

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